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# Multicritical scaling in Baxter's hard square lattice gas

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**Abstract.** The scaling behaviour of the hard square lattice gas with diagonal interactions is examined on the basis of Baxter's recent exact solution. It is demonstrated that all the corrections to scaling found may be accounted for in terms of a single irrelevant scaling field that scales simply as a length. The lattice cut-off is proposed as the source of these corrections, although other sources cannot be ruled out. The melting of the $3\times1$ commensurate ordered phase in this model is examined in a more general context in which the line of Potts critical points found by Baxter appears to be a line of *multicritical* points on which the effective uniaxially chiral symmetry breaking field vanishes.

## 1. Introduction and summary

Baxter (1980, 1981, 1982) and Baxter and Pearce (1982, 1983) have recently obtained exact expressions for the free energy, sublattice densities, correlation length and interfacial tensions of a hard square lattice gas with diagonal interactions. This exact solution contains two lines of critical points. The first separates a $3\times1$ commensurate ordered 'solid' phase from a disordered or 'fluid' phase and is apparently in the same universality class as the critical line of the three-state Potts model (e.g. Wu 1982). The second line of critical points in the exact solution (Baxter 1980, 1982) is actually the line of *tricritical* points at which the melting of the $\sqrt{2}\times\sqrt{2}$ commensurate solid phase crosses over from continuous (Ising-like) to first order, as was pointed out in a previous communication (Huse 1982) and later confirmed by further exact calculations (Baxter and Pearce 1983). In this paper, the scaling behaviour in the vicinity of these critical lines is examined.

The hard square lattice gas model considered here has three thermodynamic parameters (or fields), as is discussed in more detail below. These are the activity, $z$, and the two diagonal reduced interactions, $L$ and $M$. For each finite $L$ and $M$, Baxter (1980, 1982) has solved the model for at most one value of $z$. Thus the exact solution has been obtained only on a *two*-dimensional surface in the full *three*-dimensional $(z, L, M)$ space. Examining the exact solution in the context of this larger space was instrumental in determining that Baxter had located a line of *tricritical* points (Huse 1982). A similar examination of the line of *Potts* critical points in this larger context is undertaken below. It is concluded that this line should be *multi*critical, being, in fact, the locus on the two-dimensional boundary of the $3\times1$ commensurate phase on which the relevant uniaxially chiral symmetry breaking field vanishes (Huse and Fisher 1982, 1983). The remainder of the $3\times1$ phase boundary is expected to be in other

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universality classes than that of the three-state Potts model (Huse and Fisher 1982, 1983, Huse *et al* 1983).

The exact expressions for the free energy and sublattice densities in the vicinity of the Potts multicritical line are found to be consistent to all orders with scaling forms in which only (a) the relevant 'thermal' scaling field, (b) the marginal anisotropy field, and (c) a single irrelevant scaling field contribute. The leading scaling exponents prove to be precisely those conjectured for the three-state Potts model (den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980), values which were later derived analytically by Black and Emery (1981) and by den Nijs (1983). The non-analytic corrections to scaling found may all be accounted for by a single irrelevant scaling field that scales simply as a length squared. The spatial cut-off (the non-zero lattice constant) is proposed as the source of these corrections to scaling, although there are other possible sources that cannot be ruled out.

In the vicinity of the *tricritical* line the exact expressions for the free energy, order parameter, correlation length and interfacial tensions are, again, consistent to all orders with scaling forms in which only (a) the next-to-leading thermal scaling field, (b) the marginal anisotropy scaling field and (c) a single irrelevant scaling field contribute. For the density consistency to all orders has not yet been established, but all the corrections to scaling found to twelfth order in the natural expansion variable, $t^{1/4}$, may be accounted for by such a simple scaling form. The leading scaling exponents again agree precisely with the conjectured values, here due to Burkhardt (1980) and Nienhuis *et al* (1980). In the case of the thermal exponents, the conjectured values were later derived analytically by Nienhuis (1982). All of the non-analytic corrections to scaling revealed by the exact solutions may be attributed to a single scaling field that scales, here, as a length. The lattice cut-off is again proposed as the source of these corrections. The *leading* thermal nonlinear scaling field appears to vanish on the surface of exact solution, which in turn coincides with the surface of first-order transitions (Huse 1982). The surface of continuous (Ising-like) transitions, on the other hand, deviates from this 'scaling surface', but apparently only due to the non-vanishing correction-to-scaling field. The close proximity of this continuous transition to the surface of exact solution accounts naturally for the apparent 'non-universality' of the critical exponents found by Baxter and Pearce (1983).

## 2. The model

Baxter's (1980) exactly soluble hard square model represents a gas of particles on a planar square lattice with on-site and nearest-neighbour exclusion (thus 'hard squares') and next-nearest-neighbour interactions. It is defined on a lattice of, say, $N$ sites, each of which may be either empty or occupied; these two possibilities are represented by a discrete occupation variable, $\sigma_i$, for each site $i$, which may take on the two values $\sigma_i = 0$ or $\sigma_i = 1$. The partition function is then

$$Z_N = \sum_{\{\sigma\}} \left[ \left( \prod_i z^{\sigma_i} \right) \left( \prod_{\langle ij \rangle} (1 - \sigma_i \sigma_j) \right) \left( \prod_{\langle ijkl \rangle} \exp(L\sigma_i\sigma_k + M\sigma_j\sigma_l) \right) \right], \tag{1}$$

where the sum runs over all occupation configurations $\{\sigma\}$ and the products run over all sites, $i$, all nearest-neighbour pairs of sites, $\langle ij \rangle$, and all elementary square plaquettes, $\langle ijkl \rangle$, of the lattice, respectively. The lattice gas activity is $z$ and the reduced energies of interaction between particles connected by SW–NE and NW–SE diagonals of the

elementary squares are $L$ and $M$. A positive $L$ or $M$ represents an attractive interaction, while a negative $L$ or $M$ represents a repulsion.

Baxter (1980, 1982) has calculated exactly the partition function per site in the thermodynamic limit, namely

$$\kappa = \lim_{N\to\infty} Z_N^{1/N}, \tag{2}$$

for activities and interactions satisfying the restriction

$$z = (1-e^{-L})(1-e^{-M})/(e^{L+M}-e^{L}-e^{M}) \geqslant 0. \tag{3}$$

If, without loss of generality, we restrict our attention to $L \geqslant M$, this exact solution manifold (3) consists of two disjoint parts: the first is defined by $L \geqslant 0$ and $M \leqslant 0$ and contains Baxter's (1980) regimes I and II, whose common boundary is the line of *three-state Potts-like critical points*; the second is given by $L > 0$ and

$$e^{M} > e^{L}/(e^{L}-1) \tag{4}$$

and contains Baxter's (1980) regimes III and IV, whose common boundary is the line of *tricritical points* (Huse 1982). These two disjoint parts of Baxter's exact solution manifold are examined in turn below, with the scaling behaviour along the multicritical lines being the focus of interest.

## 3. Regimes I and II; Potts critical line

The portion of Baxter's exact solution manifold (3) with $L \geqslant 0$ and $M \leqslant 0$ is separated into two regimes by a critical line located by the equation (Baxter 1980, 1982)

$$z/(1-z\,e^{L+M})^2 = \tfrac{1}{2}(11+5\sqrt{5}), \tag{5}$$

as illustrated in figure 1. Regime I, which, overall, has weaker interactions and smaller

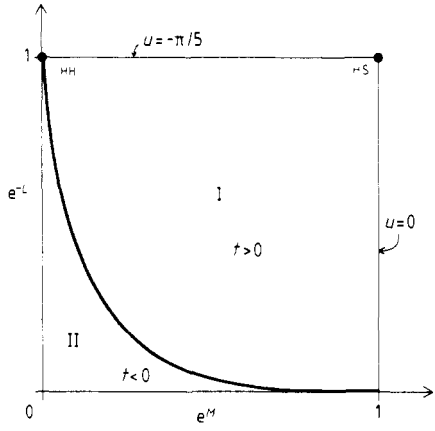


**Figure 1.** Regimes I and II and the Potts critical line (bold) projected onto a plane of constant activity $z$. For all values of the diagonal interactions $L$ and $M$ Baxter (1980, 1982) has solved the model for only one value of $z$, except in the hard hexagon (HH) limit $L \to 0$, $M \to -\infty$, where the exact solution includes all activities. For the original hard square (HS) model (e.g. Gaunt and Fisher 1965), which is $L = M = 0$, the exact solution includes only the trivial case of zero activity.

activity than regime II, represents a disordered or 'fluid' phase of the model; conversely, in regime II the lattice gas forms an ordered $3\times1$ commensurate 'solid' phase. Although the model is originally defined on a square lattice, the inequality of $L$ and $M$ in regimes I and II reduces the lattice symmetry to that of a centred rectangular lattice, with symmetry axes along the diagonals of the original square lattice. In recognition of this reduced symmetry, the ground state of the $3\times1$ ordered phase of regime II is illustrated in figure 2 with the square lattice rotated 45° from its conventional orientation.

The model defined by (1) has three parameters, namely, $z$, $L$ and $M$, and Baxter (1980) has managed to solve it only on the two-dimensional manifold (3) in the full three-dimensional $(z, L, M)$ space. It proves instructive to consider the exact solution manifold (3) in the context of this larger parameter space. A cross section for $M=-L$ is illustrated in figure 3; other cross sections at constant $M/L$ will look *qualitatively* the same. In figure 3 it is clear that the point (marked P), with Potts-like critical behaviour that Baxter (1980) has located, is just one point on the boundary of the $3\times1$ commensurate phase. Huse and Fisher (1982, 1983) have argued that the melting transition of such a $3\times1$ phase on a rectangular substrate will not generally be in the three-state Potts universality class. They suggest, based in part on study of the analogous chiral clock model (Huse *et al* 1983), that $3\times1$ melting will be in the Potts universality class *only* at a particular chemical potential (or activity) for which the

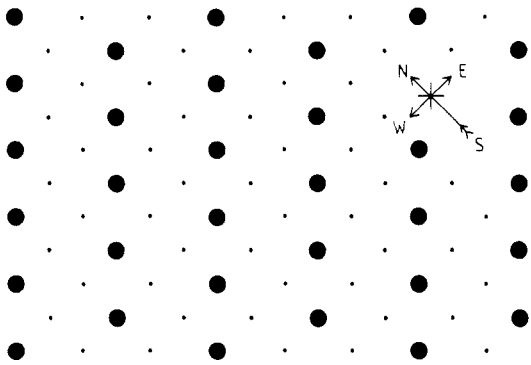


**Figure 2.** The ground state of the $3\times1$ ordering that occurs in regime II.

relevant uniaxially chiral symmetry breaking field vanishes (Huse and Fisher 1982, 1983). Thus it appears that the single point, P, on the $3\times1$ phase boundary where the model (with $M=-L$) is exactly soluble is also a *multicritical* point, the remainder of the $3\times1$ phase boundary being in different universality classes from that of the three-state Potts model. In the full $(z, L, M)$ space Baxter (1980) has, therefore, located a *line* of Potts *multi*critical points on the two-dimensional boundary of the $3\times1$ commensurate ordered phase.

Baxter (1980, 1982) has parametrised the exact solution manifold (3) in regimes I and II by the variables $u$ and $t=-q^2$, which are the arguments of the elliptic theta functions that arise in the details of the exact solution. The critical line is at $t=0$, while $t$ is positive in regime I and negative in regime II. The parameter $u$ varies from $u=-\pi/5$ on the $L=0$ axis to $u=0$ on the $M=0$ axis. The coordinate transformation relating $u$ and $t$ to $L, M$ and $z$, although complicated (equations (14.2.28–31) of Baxter 1982), is analytic both within regimes I and II and *on* the critical line.

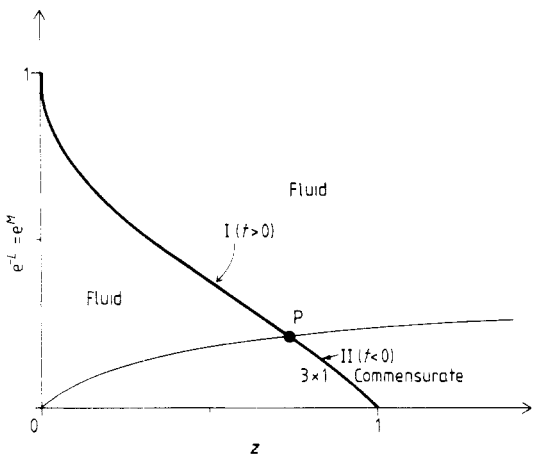


**Figure 3.** Phase diagram for $L=-M$. The line of exact solution is bold. The boundary of the $3\times1$ phase shown is schematic, except at the Potts multicritical point (P), whose location is exactly known. There may be incommensurately ordered phases between the commensurate and fluid phases at low or high activity, away from the Potts point (e.g. Huse and Fisher 1982, 1983).

The reduced free energy per site can be written as

$$f=\ln\kappa=f_a+f_s, \tag{6}$$

where $f_a$ and $f_s$ are parts analytic and singular, respectively, at the critical line. From equation (14.6.3) of Baxter (1982) we see that the singular part of the free energy may be written in the simple functional form

$$f_s=|t|^{5/3}Y_{\pm}(u,|t|^{5/3}), \tag{7}$$

where the functions $Y_{\pm}(w,x)$, applying respectively to $t\gtrless0$, consist of logarithms of elliptic theta functions and are analytic for $|x|<1$. The fact that (7) is an *equality* suggests that $t$ may be the appropriate nonlinear 'thermal' scaling field (see, e.g., Fisher 1974) for this system. The critical exponent for the free energy, $2-\alpha=\frac{5}{3}$, is precisely that expected (den Nijs 1979, Wu 1982) for a critical point in the universality class of the three-state Potts model. If (7) is regarded as a scaling form then the parameter $u$ enters as a *marginal* operator that changes the scaling function but not the critical exponents. This should not be regarded as surprising, since $u$ depends mostly on the value of the ratio $L/M$ between the diagonal interaction strengths and so may be regarded as an 'anisotropy field'. In the exact solution of the two-dimensional Ising model (e.g. McCoy and Wu 1973) the ratio of the two nearest-neighbour interaction strengths on a square lattice plays a similar role as a marginal operator. Spatial anisotropy of couplings has also been found to be marginal in $\varepsilon=4-d$ expansions (Bruce 1974).

In order to investigate the sources of the non-analytic corrections to scaling in (7), let us consider what the scaling form for the free energy in the full $(z,L,M)$ space should be. It has been argued above that the line of Potts critical points is actually *multi*critical, with the second relevant operator being uniaxially chiral symmetry breaking (Huse and Fisher 1982, 1983). Thus we should allow for a nonlinear uniaxially chiral scaling field, say $\tilde{g}_0$, as well as possible irrelevant nonlinear scaling fields,

$\tilde{g}_1, \tilde{g}_2, \ldots$. The full scaling form near the line of Potts-like multicritical points should then be

$$f_s \approx |t|^{5/3} \tilde{Y}_\pm(\tilde{g}_0/|t|^{\phi_0}, u, \tilde{g}_1|t|^{\theta_1}, \ldots). \tag{8}$$

The simplest interpretation of the behaviour (7) of $f_s$ for the exact solution manifold (3) is that on this manifold the uniaxially chiral symmetry-breaking field, $\tilde{g}_0$, vanishes and only one irrelevant scaling field, say $\tilde{g}_1$, with correction-to-scaling exponent $\theta_1 = \frac{5}{3}$, contributes to the free energy. Further, the magnitude of this single irrelevant scaling field, $\tilde{g}_1(t, u)$, must be independent of $t$.

In a renormalisation group picture, the scaling form (8) would arise from a line of fixed points parametrised by $u$ with the eigenvalues of the linearised renormalisation group along this 'fixed line' *independent* of $u$. The eigenvalue associated with the relevant 'thermal' scaling field, $t$, is $y_t = \frac{6}{5}$ (den Nijs 1979). Thus a correction-to-scaling exponent of $\theta_1 = \frac{5}{3}$ for the irrelevant scaling field $\tilde{g}_1$ would imply an associated renormalisation group eigenvalue of $y_1 = -y_t\theta_1 = -2$. Thus the simplest scenario giving rise to the confluent singularity in (7) is an irrelevant scaling field, $\tilde{g}_1$, that scales as a length squared and whose value is independent of the parameter $t$. The most likely identification of this irrelevant scaling field appears to be simply $\tilde{g}_1 = a^2$, where $a$ is the lattice spacing or spatial cut-off.

Of course, other terms in the full scaling function (8) could contribute to the confluent singularity observed on the exact solution manifold. Specifically, Nienhuis (1982) has predicted that there will be an irrelevant scaling field, say $\tilde{g}_2$, with correction-to-scaling exponent $\theta_2 = \frac{2}{3}$ at the three-state Potts critical point. If this scaling field had the value $\tilde{g}_2 = c_2(u)t$ on the manifold (3) of exact solution it would appear in the scaling function in its scaled form as $\tilde{g}_2|t|^{2/3} = c_2^{\pm}(u)|t|^{5/3}$, which is also consistent with (7). Similarly, the singular part of the free energy should be even in the uniaxially chiral scaling field, $\tilde{g}_0$; thus if one had the special relation $\tilde{g}_0 = c_0(u)t$ on the exact solution manifold, this would appear in the scaling function as an argument $(\tilde{g}_0/|t|^{\phi_0})^2 = c_0^2(u)|t|^{2-2\phi_0}$, which would be consistent with (7) if $\phi_0 = \frac{1}{6} \simeq 0.17$. The series estimate for this chiral crossover exponent (Huse *et al* 1983) is $\phi_0 = 0.19 \pm 0.06$.

Baxter (1980, 1981, 1982) has also obtained exact expressions (e.g., (14.6.7) of Baxter 1982) for the density and the order parameter on the manifold (3) of exact solution. Both are functions of $t$ only, being *independent* of $u$. Now the singular part of the density may be obtained from the singular part of the free energy via

$$\rho_s = z(\partial f_s/\partial z)_{L,M}. \tag{9}$$

In the hard hexagon limit, $L \to 0$, $M \to -\infty$, which is simply given by $u = -\pi/5$, $f_s$ has been obtained for all $z$ (Baxter 1980). Since $\rho_s$ is independent of $u$, we find

$$\begin{aligned}\rho_s(t) &= z(\partial f_s/\partial z)_{u=-\pi/5} = z(\partial t/\partial z)_{u=-\pi/5}(\partial/\partial t)[|t|^{5/3} Y_\pm(-\pi/5, |t|^{5/3})] \\ &= b(t)|t|^{2/3} P_\pm(|t|^{5/3}),\end{aligned} \tag{10}$$

where $b(t)$ is the analytic function

$$b(t) = z(\partial t/\partial z)_{u=-\pi/5}, \tag{11}$$

which is negative on the interval of interest, $-1 < t < 1$. The scaling function for the density is obtained from that for the free energy via

$$P_\pm(x) = \pm\tfrac{5}{3}(\partial/\partial x)[xY_\pm(-\pi/5, x)]. \tag{12}$$

In regime II ($t<0$) the system orders with the particles preferentially occupying one of the three sublattices of sites, as illustrated in figure 2. Let $\rho_A$ be the expectation value of the number of particles per site on the preferred ($A$) sublattice and let $\rho_B = \rho_C$ be the corresponding densities for the other two sublattices. The exact result for the order parameter is (Baxter 1980, 1981, 1982)

$$R = \rho_A - \rho_B = 3|t|^{1/9} Q(-t) Q(-t^5)/\sqrt{5} Q^2(|t|^{5/3}), \tag{13}$$

where the function $Q(x)$ is defined by

$$Q(x) = \prod_{n=1}^{\infty} (1-x^n) = 1 - x - x^2 + x^5 + x^7 + \ldots . \tag{14}$$

If $h$ is the sublattice symmetry breaking field conjugate to this order parameter then we have $R = (\partial f_s/\partial h)_{h=0}$. The field $h$ corresponds to a 'magnetic' field in a Potts model and so should scale as $|t|^{14/9}$ (Nienhuis *et al* 1980, Pearson 1980). Allowing for the corresponding nonlinear scaling field, $\tilde{h}$, the scaling form for the free energy should thus be

$$f_s \approx |t|^{5/3} \tilde{Y}_{\pm}(\tilde{h}/|t|^{14/9}, u, \tilde{g}_1|t|^{\theta_1}, \ldots). \tag{15}$$

The scaling form for the spontaneous order in regime II ($t<0$) immediately follows as

$$R \approx (\partial \tilde{h}/\partial h)_{h=0} |t|^{1/9} \tilde{M}_{II}(u, \tilde{g}_1|t|^{\theta_1}, \ldots), \tag{16}$$

where the scaling function $\tilde{M}_{II}$ is simply the derivative of $\tilde{Y}_-$ with respect to its first argument. Note that $(\partial \tilde{h}/\partial h)_{h=0}$ must be analytic in $t$ for $|t|<1$. The exact result (13) may in fact be written in such a scaling form as

$$R = (\partial \tilde{h}/\partial h)|t|^{1/9} M(|t|^{5/3}), \tag{17}$$

where, again, only the single irrelevant scaling field contributes.

In summary of this section, the exact results (Baxter 1980, 1981, 1982) for the free energy, density and order parameter in regimes I and II can be written in the simple scaling forms (7), (10) and (17), respectively. All of the non-analytic corrections to scaling embodied in these formulae could arise from a single irrelevant scaling field that scales as a length squared and whose magnitude is independent of $t$. A likely candidate for this irrelevant scaling field is the lattice spaced squared. However, other relevant or irrelevant scaling fields might also contribute to the non-analytic corrections to scaling if they enter with magnitudes proportional to $t$.

## 4. Regimes III and IV; tricritical line

The portion of Baxter's exact solution manifold (3) with $L>0$ and $M>0$, and, hence, *attractive* interactions along *both* diagonals, is again separated into two regimes by a critical line located by equation (5). Regime IV, with generally weaker interactions but larger activity, $z$, represents an ordered 'solid' phase, while regime III is a manifold on which the ordered solid phase may *coexist* with a disordered 'fluid' phase (Huse 1982). In regimes III and IV the ordering present is that of a $\sqrt{2}\times\sqrt{2}$ commensurate solid, where the particles preferentially occupy one of the *two* sublattices of next-nearest neighbour sites.

Baxter (1980, 1982) also parametrised regimes III and IV by the variables $u$ and, here, $t = +q^2$. The critical line is again given by $t=0$, while $t$ is positive in regime III

and negative in regime IV. The parameter $u$ varies from $u = 0$ for $L/M \to \infty$ to $u = \pi/5$ for $M/L \to \infty$. The phase diagram for $L = M$, and, thence, for $u = \pi/10$ on the line of exact solution, is shown in figure 4 (see also Huse 1982). For other values of $L/M$ the phase diagram should be qualitatively the same. From figure 4 we see that regime III represents a surface of first-order transitions between fluid and solid. The line separating regimes III and IV hence is actually a line of tricritical points at which the fluid-to-solid phase transition changes from first order to continuous. Regime IV, which is simply the analytic continuation of regime III in $(z, L, M)$ space, lies within the ordered solid phase.

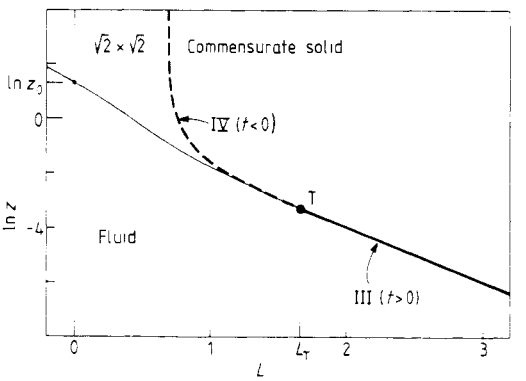


**Figure 4.** Phase diagram for $L = M$. The bold line of exact solution (regime III) coincides with the first-order solid–fluid transition for $L > L_T$, where $L_T$ locates the tricritical point, marked T. The remainder of the exact solution line (broken curve) (regime IV) lies within the solid phase. The solid–fluid transition is continuous for $L < L_T$; the Ising character of this transition has been checked only for the non-interacting case, $L = 0$, where the transition at $z_0 \simeq 3.796$ is indicated (Gaunt and Fisher 1965, Baxter *et al* 1980).

Baxter (1980, 1982) finds that the singular part of the free energy *vanishes identically* in regime III, while in regime IV it may be written in the simple form

$$f_s = |t|^{5/2} F_-(u, |t|^{5/2}), \tag{18}$$

where again the function $F_-$ consists of logarithms of elliptic theta functions and may be obtained from equations (14.6.1–3) of Baxter (1982). The variable $u$, which is still essentially the anisotropy field $L/M$, again appears as a marginal scaling field which enters only in the free energy, but not in 'observables' such as the density, order parameter, or correlation length. For the remainder of this paper it is assumed, for simplicity, that

$$L = M, \qquad u = \pi/10. \tag{19}$$

The picture for other values of $L/M$ or $u$ will be qualitatively the same.

The tricritical point separating regimes III and IV should be in the same universality class as that occurring in an *annealed* dilute Ising model (Huse 1982). The appropriate tricritical scaling exponents have been conjectured by Burkhardt (1980) and Nienhuis *et al* (1980) and the conjectures for the thermal exponents have been confirmed analytically by Nienhuis (1982). The exponent $\frac{5}{2}$ in (18) is consistent with $t$ being the next-to-leading thermal nonlinear scaling field, which has a renormalisation group

eigenvalue of $y_t = \frac{4}{5}$ (Nienhuis 1982). Allowing for the leading thermal nonlinear scaling field, $\tilde{g}$, as well as a sublattice symmetry breaking nonlinear scaling field, $\tilde{h}$, and using the calculated and conjectured exact exponents, the full scaling form near the tricritical point is expected to be

$$f_s \approx |t|^{5/2} \tilde{F}_\pm(\tilde{h}/|t|^{77/32}, \tilde{g}/|t|^{9/4}, a|t|^{5/4}, \ldots), \tag{20}$$

where $a$ again denotes the lattice constant or spatial cut-off. We will see that all of the exact results for regimes III and IV (Baxter 1980, 1981, 1982, Baxter and Pearce 1983) can be explained in terms of such a scaling form with only the four nonlinear scaling fields, $t$, $\tilde{g}$, $\tilde{h}$ and $a$, contributing and with the exact solution manifold being $\tilde{g} = \tilde{h} = 0$.

The spontaneous order at $\tilde{h} = 0$ in the solid phase is obtained from $f_s$ by differentiating with respect to the appropriate sublattice symmetry breaking field, $h$. This results in the scaling expression

$$R = (\partial f_s/\partial h)_{h=0} \approx (\partial \tilde{h}/\partial h)|t|^{3/32} \tilde{M}_\pm(\tilde{g}/|t|^{9/4}, a|t|^{5/4}, \ldots), \tag{21}$$

where $\tilde{M}_\pm$ is the derivative of $\tilde{F}_\pm$ with respect to its first argument. For regime III ($t > 0$ and $\tilde{g} = 0$) Baxter and Pearce (1983) find

$$R(t \geqslant 0) = (\tfrac{8}{5})^{1/2} t^{3/32} Q(t) Q(t^5) / Q(t^{5/4}) Q(t^{5/2}), \tag{22}$$

where $Q(x)$ is as defined by (14). This is certainly consistent with (21), which, for the exact solution line $\tilde{g} = 0$, may evidently be written as the *equality*

$$R = (\partial \tilde{h}/\partial h)|t|^{3/32} M_\pm(|t|^{5/4}). \tag{23}$$

If we now assume that the scaling function $M_+(x)$ is analytic at the origin we may make the identifications

$$(\partial \tilde{h}/\partial h)(t) = Q(t) C(t^5) \tag{24}$$

and

$$M_+(x) = (\tfrac{8}{5})^{1/2} Q(x^4) / Q(x) Q(x^2) C(x^4) \tag{25}$$

where $C(w)$ is some analytic function on the interval $-1 < w < 1$ with $C(0) = 1$.

For the spontaneous order in regime IV Baxter and Pearce (1983) find

$$R = (\tfrac{4}{5})^{1/2} |t|^{1/4} Q(t) Q^2(t^{10}) / Q(t^5) Q^2(-t^5). \tag{26}$$

Comparing this with (23) and (24) we find

$$M_-(x) = (\tfrac{4}{5})^{1/2} x^{1/8} Q^2(x^8) / Q(-x^4) Q^2(x^4) C(-x^4), \tag{27}$$

which is, at first sight, surprising, since one does not generally expect singularities in the scaling function. However, from figure 4 it is apparent that there is an Ising-like critical line very near regime IV. This critical line will be contained in the full scaling function, $\tilde{M}_-$, which, due to the Ising critical exponent $\beta = \beta_I = \frac{1}{8}$ (e.g. McCoy and Wu 1973), should behave as

$$\tilde{M}_-(w, x, \ldots) \approx |w - w_c(x, \ldots)|^{1/8} \tilde{A}(w, x, \ldots), \tag{28}$$

where $\tilde{A}$ is non-zero at the transition which is located at $w = w_c$. On recalling that

$$M_-(x) = \tilde{M}_-(0, x, \ldots) \tag{29}$$

and assuming that $\tilde{A}(0, x, \ldots)$ does not vanish for $x \to 0$, we find, by comparing (27)

and (28), that the location of the Ising-like transition behaves as

$$w_c(x, \ldots) \sim x. \tag{30}$$

for $x \to 0$.

Returning to our original parametrisation via $w = \tilde{g}|t|^{-9/4}$ and $x = a|t|^{5/4}$ we see that the Ising-like critical line behaves as

$$\tilde{g}_c \sim |t|^{7/2} \tag{31}$$

for $t \to 0^-$. Since $t$ measures distance from the tricritical point along the exact solution line and $\tilde{g}$ measures deviation from the exact solution line in the $(z, L)$ plane (see figure 4), this means that the critical line possesses a very high degree of tangency to regime IV of the exact solution line. In fact, the above analysis says that the deviation of the critical line from regime IV, which serves as the scaling axis, is not contained in the *leading* scaling function, but is due only to the corrections to scaling arising from the lattice cut-off. This explains why the order parameter exponents (and, see below, the correlation length exponents) are not equal in regimes III and IV as would generally be expected.

The exact results for the densities in regimes III and IV are strikingly similar. Baxter and Pearce (1983) find that in regime IV

$$\rho^{IV}(t) = D(t), \tag{32}$$

while in regime III

$$\rho^{III}_{sol}(t) = D(-t^{1/4}), \tag{33}$$

$$\rho^{III}_{fl}(t) = D(t^{1/4}), \tag{34}$$

for the solid and fluid phases. The function $D$ is defined by

$$D(x) = H_1(x)H_1(x^4)/P^2(-x^5), \tag{35}$$

where the functions $H_1$ and $P$ are, in turn, defined by (14.6.5*f*) and (14.6.5*d*) of Baxter (1982). The function $D(x)$ is analytic on $-1 < x < 1$ and has the expansion

$$D(x) = \rho_c - 5^{-1/2}[x + x^4 - x^5 - 2x^6 - x^9 + 2x^{10} + 4x^{11} + O(x^{13})], \tag{36}$$

where $\rho_c = \frac{1}{10}(5 - \sqrt{5})$ is the multicritical density. (Baxter and Pearce (1983) expanded $D(x)$ to order $x^4$ in their equations (4.30), (4.31) and (4.36), but are in error in the coefficient of $x^4$. This error was brought to my attention by the following scaling argument.)

The singular part of the density is obtained, via $\rho_s = z(\partial f_s/\partial z)$, from the singular part of the free energy. If we assume, as above, that only those scaling fields exhibited in (20) contribute, we find

$$\rho_s = z(\partial t/\partial z)(\partial f_s/\partial t) + z(\partial \tilde{g}/\partial z)(\partial f_s/\partial \tilde{g}). \tag{37}$$

Since $(\partial f_s/\partial t) \sim |t|^{3/2}$ and $(\partial f_s/\partial \tilde{g}) \sim |t|^{1/4}$, with only $|t|^{5/4}$ corrections, the terms in the density analytic in $t$ through order $t^3$ must arise from the analytic part of the density, which is therefore

$$\rho_a = \rho_c - 5^{-1/2}t + O(t^4). \tag{38}$$

In regime III, $(\partial f_s/\partial t)$ vanishes identically, so that the scaling form for the singular part of the density should be

$$\rho_s = z(\partial \tilde{g}/\partial z)t^{1/4}\Delta_{sol,fl}(t^{5/4}), \tag{39}$$

for the solid and fluid phases. By comparing (33) and (34) we see that the scaling functions are related by the equation

$$\Delta_{\mathrm{sol}}(x) = -\Delta_{\mathrm{fl}}(-x). \tag{40}$$

The exact expressions for $\rho$ are not in a form in which (39) can be shown to be adequate to all orders in $t$, nonetheless, to the order shown in (36) the singular part of the density of the solid phase in regime III may be written as

$$\rho_{\mathrm{s}} = (1-t-t^2)t^{1/4}5^{-1/2}(1+2t^{5/4}+4t^{5/2}) + \mathrm{O}(t^{13/4}). \tag{41}$$

This is certainly consistent with the simple scaling form (39). Perhaps (39) can be verified to all orders by someone more familiar with the properties of the elliptic functions $H_1$ and P. (*Note added in proof.* Baxter (private communication) has now done this.)

Baxter and Pearce (1983) have also obtained the exact behaviour of the correlation length in regimes III and IV. They studied the so-called 'true correlation length', $\xi_0$, defined by

$$\langle\sigma_0\sigma_l\rangle - \langle\sigma_0\rangle\langle\sigma_l\rangle \sim \mathrm{e}^{-l/\xi_0}(A + B\,\mathrm{e}^{-\mathrm{i}\pi l}), \tag{42}$$

where the occupation variables $\sigma_0$ and $\sigma_l$ refer to two sites $l$ lattice units apart in the same column or row. This correlation length is found to be simply proportional to the inverse interfacial tension in both regimes (Baxter and Pearce 1983). The correlation length should scale with the same variables as the free energy, so since $\nu = \frac{5}{4}$ (e.g. Nienhuis 1982), we expect

$$\xi_0^{-1} \approx |t|^{5/4}\tilde{X}_{\pm}(\tilde{g}/|t|^{9/4}, a|t|^{5/4}\ldots). \tag{43}$$

The exact results (Baxter and Pearce 1983) are indeed found to be consistent with such a scaling form, with, in fact, no dependence on $u$, and may be written as the *equality*

$$\xi_0^{-1} = |t|^{5/4}X_{\pm}(|t|^{5/4}), \tag{44}$$

where the scaling functions $X_{\pm}$ are again simply related to the logarithms of certain elliptic functions (see §§ 2 and 3 of Baxter and Pearce 1983).

For $t > 0$ (regime III) the scaling function $X_+(x)$ does not vanish for $x \to 0$ so one has $\xi_0 \sim t^{-5/4}$. However, in regime IV, for $t < 0$, the scaling function $X_-(x)$ must again vanish as $x \to 0$ in order to reflect the nearby Ising critical line, as did $M_-(x)$ above. Now, the inverse correlation length vanishes *linearly* at the Ising transition (e.g. McCoy and Wu 1973) so we have

$$\tilde{X}_-(w, x, \ldots) \sim |w - w_{\mathrm{c}}(x, \ldots)| \tag{45}$$

or $X_-(x) \sim x$, assuming, as before, that $w_{\mathrm{c}}(x, \ldots) \sim x$ (equation (30)). Thus we obtain

$$\xi_0 \sim |t|^{5/2} \tag{46}$$

for $t \to 0^-$ in regime IV, in agreement with the exact results (Baxter and Pearce 1983).

As a final point, it seems noteworthy that all the corrections to scaling near the Potts-like critical line may be attributed to the lattice cut-off *squared*, while near the tricritical line there are corrections that, by the above arguments, are found to be *linear* in the lattice cut-off. Corrections scaling as the cut-off squared are not surprising; for example, when a $\phi^4$ model on a lattice is mapped onto a continuum $\phi^4$ model, a term in the Hamiltonian proportional to $a^2(\nabla^2\phi)^2$ is found that will certainly give such corrections for dimensionalities $d > 4$. Corrections linear in the lattice cut-off are not

so easily explained. However, there is one other exactly soluble model in which they appear to occur, namely the eight-vertex model (e.g. Baxter 1982). The exact result for the magnetisation of the eight-vertex model (Baxter 1982) may be written as

$$M = \sqrt{2}t^{\beta}Q(t^{4\nu})/Q(-t^{\nu}), \tag{47}$$

where $t$, as in the hard square models, appears to be the nonlinear scaling field for this problem and is related to the couplings via elliptic functions (Baxter 1982). The correlation length exponent, $\nu$, and the order parameter exponent, $\beta$, are related by

$$\nu = 8\beta \tag{48}$$

and vary continuously along the critical line of this model. If the non-analytic corrections to scaling in (47) are all attributed to the lattice cut-off, then the leading corrections are clearly *linear* in the cut-off. Since the Ising model is just a special case (namely, $\nu = 1$) of the eight-vertex model, this seems to contradict the suggestion of Aharony and Fisher (1980) that corrections linear in the cut-off do not occur for the planar Ising models.

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